Focusing on the Fundamentals of Math

A TEACHER'S GUIDE

This guide is intended to support teachers' ongoing efforts in building students' knowledge and skills in mathematics. It focuses attention on the content of expectations in *The Ontario Curriculum, Grades 1–8: Mathematics, 2005* that deal with fundamental mathematics concepts and skills (specifically, expectations in the Number Sense and Numeration strand and expectations that relate to number properties in the Patterning and Algebra strand). The guide outlines steps to achieving the knowledge and skills described in these expectations and suggests how to make more timely connections that will better support student learning. A strong foundation in the concepts and skills emphasized here will prepare students for success in high school, and ensure that they have a set of essential skills for employment and responsible citizenship in the future.

> Becoming highly skilled at arithmetic requires the development of number sense alongside procedural and factual knowledge as well as the mathematical principles that govern how the operations are related to one another.
>
> (Bruce & Chang, 2013, p. 14, citing Baroody & Dowker, 2003)

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What are the fundamental math concepts and skills?

Fundamental math skills, and the concepts that underpin them, may be categorized according to the following framework:

- **Working with numbers:** Understanding and using numbers (e.g., being able to read, represent, count, order, estimate, compare, compose, decompose, and recompose numbers).
- **Recognizing and applying understanding of number properties:** Understanding how numbers behave in operations and drawing on that understanding to master math facts and perform calculations.
- **Mastering math facts:** Understanding and recalling math facts, using a variety of strategies.
- **Developing mental math skills:** Doing calculations in the mind, with little or no use of paper and pencil or calculator.
- **Developing proficiency with operations:** Performing calculations with ease, precision, and consistency and with a general understanding of number and operations, number properties, and their appropriate application in problem solving.

Why is it important for students to master the fundamentals of math?

Understanding how numbers work is foundational to all aspects of mathematics. As students progress through the grades, they learn about different types of numbers and how those numbers behave when operations are applied to them. Recognizing and understanding number properties is foundational to arithmetic and algebra.

Students need to be fluent with number facts in order to perform mathematical calculations efficiently and accurately, whether mentally or by applying algorithms on paper. The goal is for students to develop **automaticity**, which is the ability to use skills or perform mathematical procedures with little or no mental effort. Automaticity with math facts also supports students in critical thinking and problem solving.

> The more automatically a procedure can be executed, the less mental effort is required. Since each person has a limited amount of mental effort that he or she can expend at any one time, more complex tasks can be done well only when some of the subtasks are automatic.

> (National Research Council, 2001, p. 351)

Most students learn math facts gradually, over a number of years, using tools such as manipulatives and calculators. Mastery comes with practice, and practice helps consolidate knowledge. Students will draw on their ability to apply math facts with automaticity throughout secondary school, as they manipulate algebraic expressions and equations.

Mental math skills involve the ability to perform mathematical calculations in the mind, without relying on pencil and paper. Mental math skills enable students to estimate

answers to calculations, and so be able to work quickly on everyday problems and judge the reasonableness of answers calculated formally.

It is important for students to become proficient in using the operations of addition, subtraction, multiplication, and division in the elementary grades. Even in today's technological age, people use calculations every day – for example, to verify that they've received correct change or to estimate how many cans of paint they need to paint a room. In the early grades, students learn about operations with whole numbers, and this sets the stage for working with decimals, fractions, and integers later on.

Though individual students may progress at different rates, generally speaking, addition/ subtraction facts should be mastered by the end of Grade 3, and multiplication/division facts should be mastered by the end of Grade 5 (Chapin & Johnson, 2006) – but students should continue to practise and extend their proficiency throughout the grades and in the context of learning in all the strands of the mathematics curriculum.

How can educators help students master the fundamentals?

> Strategies help students find an answer even if they forget what was memorized. Discussing math fact strategies focuses attention on number sense, operations, patterns, properties, and other critical number concepts.
>
> (O'Connell & SanGiovanni, 2011, p. 5)

Fluency with basic math facts is fostered through instruction that *highlights strategies for remembering facts, focuses on making sense, and integrates math-fact learning into other aspects of math learning*, such as developing computational skills. Repeated practice, or "drill", by itself may improve speed, but it does not contribute to understanding and it is not sufficient to guarantee immediate recall. Strategies such as learning related math facts together – for instance, "× 5 is half of × 10" – enable students to understand the inter-connectedness of math facts and also make it easier to remember them.

> Children should learn their number facts. However, they would benefit from learning these facts by using an increasingly sophisticated series of strategies rather than by jumping directly to memorization.
>
> (Lawson, 2016, p. 4)

Strategies that can help students commit basic facts to memory include:

- identifying the various ways in which math facts can be understood, such as "+ 1", "- 1", "+ 2", "- 2", "+ 10", "- 10", "doubles", "making 10", "× 2 or doubling", "× 10", "× 5";
- learning about number properties as they notice patterns in addition and multiplication – for example, as students in the primary grades realize that two numbers added backward or forward give the same sum, they learn about the commutative property of addition;

- looking for relationships between numbers – for example: 3×5 is the same as $5 + 5 + 5$, or half of 30, or one more than $7 + 7$, or halfway between 10 and 20;
- representing operations performed on numbers in different ways – for example, in concrete, pictorial, and symbolic ways;
- analysing results of operations – for example, "What happens when two odd numbers are added?";
- practising by playing games, such as Domino Drop, Roll-O, and Over-Easy Doubles;[1] and
- having frequent opportunities to practise.

Educators' observations and their conversations with students provide them with rich insight into the strategies that students are using and how effectively they are applying them. Conversations reveal whether students understand how they are performing computations and whether their answers make sense to them. Research shows that, for many students, timed testing may be less constructive, as it fosters math anxiety, which negatively impacts the students' efficiency and accuracy.

As educators plan student learning experiences, it is important to focus on student understanding and sense making, the interconnectedness of the categories, and the application of skills in problem-solving contexts both in and outside the classroom. The goal should be to provide opportunities for students to come to recognize, informally, how numbers and operations work. Only then should formal methods, such as algorithms, be introduced, modelled, and supported.

How can educators use this guide to support students in learning the fundamental math concepts and skills?

The following tables, for Grades 1 to 3, Grades 4 to 6, and Grades 7 and 8, outline a "scope and sequence", from *The Ontario Curriculum, Grades 1–8: Mathematics, 2005*, for developing and mastering the fundamentals, based on the framework outlined on page 2 of the present document. As educators support students in meeting the curriculum expectations, they help them master the skills and knowledge indicated in the tables for each grade by the end of the school year, giving consideration to individual students' learning needs.

The fundamental concepts and skills outlined in the following tables can be developed in connection with learning in all strands of the math curriculum – Number Sense and Numeration, Geometry and Spatial Sense, Measurement, Data Management and Probability, and Patterning and Algebra.

> In order to become fluent in calculation, students must have efficient, accurate methods supported by number and operation sense. They must learn how algorithms work.
>
> (Sutton & Krueger, 2002, p. 82)

1. See Ministry of Education, *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6. Volume 5: Teaching Basic Facts and Multidigit Computations*, Appendix 10-2, n.d., p. 69.

MATH FUNDAMENTALS IN GRADES 1, 2, AND 3

CATEGORY*	GRADE 1	GRADE 2	GRADE 3

Working with Numbers	• Understand and use:
	  ◆ whole numbers to 50 (i.e., 0, 1, 2, 3, … 50)
	  ◆ anchors of 5 and 10
	  ◆ fractions, as follows: divide whole objects into equal-sized parts and identify the parts as unit fractions, e.g., $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, using various models, such as an area model, number line model, volume model, set model
		• Understand and use:
		  ◆ whole numbers to 100 (i.e., 0, 1, 2, 3, … 100)
		  ◆ fractions, as follows: compose and decompose wholes using unit fractions, e.g., show that $\frac{9}{4}$ is the same as two wholes and one-fourth; compare and order unit fractions using various models
			• Understand and use:
			  ◆ whole numbers to 1000 (i.e., 0, 1, 2, 3, … 1000)
			  ◆ fractions, as follows: divide whole objects and sets of objects into equal parts, and identify parts using fractional names

Recognizing and Applying Understanding of Number Properties	• Recognize the:
	  ◆ property of zero in addition, i.e., $a + 0 = a$
	  ◆ property of zero in subtraction, i.e., $a - 0 = a$
	  ◆ commutative property of addition, i.e., $a + b = b + a$
	• **Apply understanding of number properties in doing calculations**
		• **Apply understanding of number properties in doing calculations**	• Recognize the:
			  ◆ commutative property of multiplication, i.e., $a \times b = b \times a$
			  ◆ property of zero in multiplication, i.e., $a \times 0 = 0$
			  ◆ property of one in multiplication, i.e., $a \times 1 = a$
			  ◆ associative property of addition, i.e., $(a + b) + c = a + (b + c)$
			• **Apply understanding of number properties in doing calculations**

Mastering Math Facts	• Understand and recall math facts for:
	  ◆ addition to 10, and related subtraction
		• Understand and recall math facts for:
		  ◆ addition to 20, and related subtraction
			• Understand and recall math facts for:
			  ◆ multiplication from 0×0 to 7×7, and related division

Developing Mental Math Skills	• Describe and use strategies to:
	  ◆ add and subtract whole numbers to 10
	  ◆ estimate the number of objects in a set, and check by counting, e.g., by grouping objects into 5s or 10s
		• Describe and use strategies to
		  ◆ add and subtract whole numbers to 20
			• Describe and use strategies to:
			  ◆ add and subtract two-digit whole numbers
			  ◆ multiply to 7×7
			  ◆ divide to $49 \div 7$
			  ◆ round two-digit whole numbers to the nearest ten
			  ◆ estimate when solving problems involving addition and subtraction

Developing Proficiency with Operations	• Add and subtract whole numbers to 20 • Add and subtract money amounts to 20¢, using coin manipulatives and drawings • **Recognize the inverse relationship between addition and subtraction (e.g., since 4 + 5 = 9, then 9 – 5 = 4) and apply this understanding in doing calculations**	• Add and subtract whole numbers to 100 • Add and subtract money amounts to 100¢ • **Recognize the inverse relationship between addition and subtraction and apply this understanding in doing calculations** • Represent multiplication as the combining of equal groups, that is, as repeated addition (e.g., use counters to show that 3 groups of 2 is equal to 2 + 2 + 2 and to 3 × 2) • Represent division as the sharing of a quantity equally (e.g., "I can share 12 carrot sticks equally among 4 friends by giving each person 3 carrot sticks.")	• Add and subtract whole numbers to 1000 • Add and subtract money amounts to make simulated purchases and change for amounts up to $10 • **Recognize the inverse relationship between addition and subtraction and apply this understanding in doing calculations** • **Recognize the inverse relationship between multiplication and division (e.g., since 4 × 5 = 20, then 20 ÷ 5 = 4) and apply this understanding in doing calculations**

*See the discussion of the categories on page 2.

Notes:
- These tables are provided strictly as a resource for teachers, to assist in focusing in on the key foundational math concepts and skills embedded in the curriculum expectations in *The Ontario Curriculum, Grades 1–8: Mathematics, 2005*. For purposes of instruction and evaluation, mandatory learning is as described in the curriculum expectations.

- Blue, boldface font indicates items practised constantly, from grade to grade.

MATH FUNDAMENTALS IN GRADES 4, 5, AND 6

CATEGORY*	GRADE 4	GRADE 5	GRADE 6
Working with Numbers	• Understand and use: ◆ whole numbers to 10 000 (i.e., 0, 1, 2, 3, … 10 000) ◆ decimal numbers to tenths ◆ fractions, as follows: compare and order fractions with like numerators by considering the size and the number of fractional parts or by using benchmarks of 0, $\frac{1}{2}$ and 1; demonstrate and explain the relationship between equivalent fractions, using concrete materials and drawings ◆ relationships between fractions and decimals to tenths	• Understand and use: ◆ whole numbers to 100 000 (i.e., 0, 1, 2, 3, … 100 000) ◆ decimal numbers to hundredths ◆ fractions, as follows: compare and order fractions with like denominators, including proper and improper fractions and mixed numbers; demonstrate and explain the concept of equivalent fractions, using concrete materials ◆ relationships between fractions and their equivalent decimal forms	• Understand and use: ◆ whole numbers to 1 000 000 (i.e., 0, 1, 2, 3, … 1 000 000) ◆ decimal numbers to thousandths ◆ fractions, as follows: compare and order fractions with unlike denominators, including proper and improper fractions and mixed numbers ◆ relationships among fractions, decimals, and percents ◆ composite and prime numbers, and the relationship between them
Recognizing and Applying Understanding of Number Properties	• Recognize the: ◆ associative property of multiplication, i.e., (a × b) × c = a × (b × c) ◆ distributive property of multiplication over addition, i.e., $a \times (b + c) = (a \times b) + (a \times c)$ • **Apply understanding of number properties in doing calculations**	• **Apply understanding of all number properties of addition and multiplication in doing calculations**	• **Apply understanding of all number properties of addition and multiplication in doing calculations**
Mastering Math Facts	• Understand and recall math facts for: ◆ multiplication from 0 × 0 to 10 × 10, and related division	• **Continue to practise math facts and begin to apply knowledge automatically in doing calculations**	• **Continue to practise math facts and practise applying knowledge automatically in doing calculations**
Developing Mental Math Skills	• Describe and use strategies to: ◆ add and subtract two-digit numbers ◆ multiply to 10 × 10 ◆ divide to 100 ÷ 10 ◆ multiply whole numbers by 10, 100, and 1000, and divide by 10 and 100 ◆ round four-digit numbers to the nearest ten, hundred, and thousand ◆ estimate addition, subtraction, and multiplication of whole numbers	• Describe and use strategies to: ◆ add, subtract, and multiply whole numbers ◆ multiply whole numbers to 10, 100, 1000, and 10 000, and divide decimal numbers by 10 and 100 ◆ round decimal numbers to the nearest tenth ◆ estimate addition, subtraction, multiplication, and division of whole numbers	• Describe and use strategies to: ◆ add, subtract, multiply, and divide whole numbers ◆ multiply whole numbers by 0.1, 0.01, and 0.001 ◆ multiply and divide decimal numbers by 10, 100, 1000, and 10 000 ◆ estimate addition and subtraction of whole numbers and decimals
Developing Proficiency with Operations	• Add and subtract whole numbers to 10 000 • Add and subtract decimal numbers to tenths • Add and subtract money amounts to make simulated purchases and change for amounts up to $100 • **Recognize the inverse relationship between addition and subtraction and apply this understanding in doing calculations** • Multiply two-digit whole numbers by one-digit whole numbers • Divide two-digit whole numbers by one-digit whole numbers • Describe simple whole-number multiplicative relationships, including those involving unit rates • **Recognize the inverse relationship between multiplication and division and apply this understanding in doing calculations**	• Add and subtract decimal numbers to hundredths, including money amounts • **Recognize the inverse relationship between addition and subtraction and apply this understanding in doing calculations** • Multiply two-digit whole numbers by two-digit whole numbers • Divide three-digit whole numbers by one-digit whole numbers • Describe multiplicative relationships between quantities by using simple fractions and decimals • Demonstrate an understanding of proportional reasoning using simple multiplicative relationships involving whole number rates • **Recognize the inverse relationship between multiplication and division and apply this understanding in doing calculations**	• Add and subtract decimal numbers to thousandths • **Recognize the inverse relationship between addition and subtraction and apply this understanding in doing calculations** • Multiply and divide decimal numbers to tenths by whole numbers • Multiply and divide whole numbers (four-digit by two-digit) • Demonstrate an understanding of proportional reasoning using ratios and unit rates • **Recognize the inverse relationship between multiplication and division, and apply this understanding in doing calculations** • Explain the need for a standard order for performing operations, and use the order for calculations with whole numbers

MATH FUNDAMENTALS IN GRADES 7 AND 8

CATEGORY*	GRADE 7	GRADE 8

Working with Numbers	• Understand and use:
	  ◆ whole numbers (no ceiling)
	  ◆ decimal numbers (no ceiling)
	  ◆ fractions (positive)
	  ◆ integers (no ceiling)
	  ◆ multiples and factors of whole numbers
	  ◆ perfect squares and square roots
	  ◆ rate as a comparison, or ratio, of two measurements with different units
	  ◆ relationships among fractions, decimals, percents, and ratios
		• Understand and use:
		  ◆ rational numbers (whole numbers, including in expanded form; integers; positive and negative fractions; and decimals to thousandths)
		  ◆ exponents
		  ◆ common factors and common multiples
		  ◆ paired quantities that are directly proportional, including ratios and rates, identified in real-life situations (e.g., the number of servings and the quantities in a recipe, mass and volume of a substance, circumference and diameter of a circle)
		  ◆ translations between equivalent forms of a number, i.e., fractions, decimals, and percents

Recognizing and Applying Understanding of Number Properties	• **Apply understanding of all number properties of addition and multiplication in doing calculations**	• **Apply understanding of all number properties of addition and multiplication in doing calculations**
Mastering Math Facts	• **Continue to practise math facts and apply knowledge automatically in doing calculations**	• **Continue to practise math facts and apply knowledge automatically in doing calculations**

Developing Mental Math Skills	• Describe and use strategies to:
	  ◆ generate multiples and factors
	  ◆ represent perfect squares (to 10^2) and square roots (to 100), using a variety of tools (e.g., geoboards, connecting cubes, grid paper)
	  ◆ solve problems involving the addition and subtraction of fractions and decimals
	  ◆ estimate operations with whole numbers, decimals, and percents
		• Describe and use strategies to:
		  ◆ estimate operations with whole numbers, decimals, percents, integers, and fractions
		  ◆ estimate, and verify using a calculator, the positive square roots of whole numbers

Developing Proficiency with Operations	• Multiply and divide decimal numbers to thousandths by one-digit whole numbers
	• Divide whole numbers by simple fractions and by decimal numbers to hundredths
	• Evaluate expressions that involve whole numbers and decimals, including expressions that contain brackets, using order of operations
	• Add and subtract fractions with simple like and unlike denominators
	• Demonstrate, using concrete materials, the relationship between the repeated addition of fractions and the multiplication of that fraction by a whole number
	• Add and subtract integers
	• Solve problems involving:
	  ◆ whole numbers
	  ◆ decimals
	  ◆ fractions
	  ◆ integers
	  ◆ percents
	  ◆ unit rates
		• Multiply and divide decimal numbers by various powers of ten
		• Solve multi-step problems involving whole numbers and decimals
		• Represent the multiplication and division of fractions
		• Evaluate expressions that involve integers, including expressions that contain brackets and exponents, using order of operations
		• Add, subtract, multiply, and divide simple fractions
		• Solve problems involving operations with integers
		• Express repeated multiplication using exponential notation
		• Solve problems involving percents expressed to one decimal place and whole-number percents greater than 100
		• Solve problems involving proportions, rates, and ratios

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